Appendix A

Definitions for Ordered Sets

The appendices contain all the formal definitions, propositions and proofs for developing a model of the display process based on lattices. Here we list some basic definitions from the theory of ordered sets.

Def. A *partially ordered set* (*poset*) is a set D with a binary relation $\leq$ on D such that, $\forall x, y, z \in D$

(a) $x \leq x$ "reflexive"

(b) $x \leq y$ & $y \leq x \Rightarrow x = y$ "anti-symmetric"

(c) $x \leq y$ & $y \leq z \Rightarrow x \leq z$ "transitive"

Def. An *upper bound* for a set $M \subseteq D$ is an element $x \in D$ such that $\forall y \in M.\ y \leq x$.

Def. The *least upper bound* of a set $M \subseteq D$, if it exists, is an upper bound x for M such that if y is another upper bound for M, then $x \leq y$. The least upper bound of M is denoted *sup* M or $\bigvee M$. $sup\{x,y\}$ is written $x \vee y$.

Def. A *lower bound* for a set $M \subseteq D$ is an element $x \in D$ such that $\forall y \in M.\ x \leq y$.

Def. The *greatest lower bound* of a set $M \subseteq D$, if it exists, is a lower bound x for M such that if y is another lower bound for M, then $y \leq x$. The greatest lower bound of M is denoted *inf* M or $\bigwedge M$. $inf\{x,y\}$ is written $x \wedge y$.

Def. A subset $M \subseteq D$ is a *down set* if $\forall x \in M. \forall y \in D.\ y \leq x \Rightarrow y \in M$. Given $M \subseteq D$, define $\downarrow M = \{y \in D \mid \exists x \in M.\ y \leq x\}$, and given $x \in D$, define $\downarrow x = \{y \in D \mid y \leq x\}$.

Def. A subset $M \subseteq D$ is an *up set* if $\forall x \in M.\ \forall y \in D.\ x \leq y \Rightarrow y \in M$. Given $M \subseteq D$, define $\uparrow M = \{y \in D \mid \exists x \in M.\ x \leq y\}$, and given $x \in D$, define $\uparrow x = \{y \in D \mid x \leq y\}$.

Def. A subset $M \subseteq D$ is a *chain* if, for all $x, y \in M$, either $y \leq x$ or $x \leq y$.

Def. A subset $M \subseteq D$ is *directed* if, for every finite subset $A \subseteq M$, there is an $x \in M$ such that $\forall y \in A.\ y \leq x$.

Def. A *poset* D is *complete* (and called a *cpo*) if every directed subset $M \subseteq D$ has a least upper bound $\bigvee M$ and if there is a least element $\perp \in D$ (that is, $\forall y \in D.\ \perp \leq y$).

Def. If D and E are *posets*, we use the notation $(D \rightarrow E)$ to denote the set of all functions from D to E.

Def. If D and E are *posets*, a function $f{:}D \rightarrow E$ is *strict* if $f(\perp) = \perp$.

Def. If D and E are *posets*, a function $f{:}D \to E$ is *monotone* if $\forall x, y \in D.\ x \leq y \Rightarrow f(x) \leq f(y)$. We use the notation $MON(D \to E)$ to denote the set of all monotone functions from D to E.

Def. If D and E are *posets*, a function $f{:}D \to E$ is an *order embedding* if $\forall x, y \in D.\ x \leq y \Leftrightarrow f(x) \leq f(y)$.

Def. Given *posets* D and E, a function $f{:}D \to E$, and a set $M \subseteq D$, we use the notation $f(M)$ to denote $\{f(d) \mid d \in M\}$.

Def. If D and E are *cpos*, a function $f{:}D \to E$ is *continuous* if it is monotone and if $f(\bigvee M) = \bigvee f(M)$ for directed $M \subseteq D$.

Def. If D is a *cpo*, then $x \in D$ is *compact* if, for all directed $M \subseteq D$, $x \leq \bigvee M \Rightarrow \exists y \in M.\ x \leq y$.

Def. A *cpo* D is *algebraic* if for all $x \in D$, $M = \{y \in D \mid y \leq x \ \&\ y \text{ compact}\}$ is directed and $x = \bigvee M$.

Def. A *cpo* D is a *domain* if D is algebraic and if D contains a countable number of compact elements.

Most of the ordered sets used in programming language semantics are domains.

Def. A *poset* D is a *lattice* if for all $x, y \in D$, $x \vee y$ and $x \wedge y$ exist in D.

Def. A *poset* D is a *complete lattice* if for all $M \subseteq D$, $\bigvee M$ and $\bigwedge M$ exist in D.

Def. If D and E are lattices, a function $f{:}D{\to}E$ is a *lattice homomorphism* if for all $x, y \in D$, $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$. If $f{:}D{\to}E$ is also a bijection then it is a *lattice isomorphism*.

Def. A binary relation $\equiv$ on a set D is an *equivalence relation* if $\forall x, y, z \in D$

(a) $x \equiv x$ "reflexive"

(b) $x \equiv y \Leftrightarrow y \equiv x$ "symmetric"

(c) $x \equiv y$ & $y \equiv z \Rightarrow x \equiv z$ "transitive"

Def. id_D denotes the identity function on D. Given a function $f{:}D{\to}D$, $im(f) = \{f(d) \mid d \in D\}$.

Def. If D is a *cpo*, a continuous function $f{:}D{\to}D$ is a *retraction* of D if $f = f \circ f$. A retraction $f{:}D{\to}D$ is a *projection* if $f \leq id_D$ and a *finitetary projection* if in addition $im(f)$ is a domain. A retraction $f{:}D{\to}D$ is a *closure* if $f \geq id_D$ and a *finitetary closure* if in addition $im(f)$ is a domain.

Def. If D and E are *cpos*, a pair of continuous functions $f{:}D{\to}E$ and $g{:}E{\to}D$ are a *retraction pair* if $g \circ f \leq id_D$ and $f \circ g = id_E$. The function g is called an *embedding*, and f is called a *projection*.

Appendix B

Proofs for Section 3.1.4

Here we present the technical details for Section 3.1.4. We can interpret Mackinlay's expressiveness conditions as follows:

Condition 1. $\forall P \in MON(U \rightarrow \{\bot , 1\}).\ \exists Q \in MON(V \rightarrow \{\bot , 1\}).$
$\forall u \in U.\ P(u) = Q(D(u)).$

Condition 2. $\forall Q \in MON(V \rightarrow \{\bot , 1\}).\ \exists P \in MON(U \rightarrow \{\bot , 1\}).$
$\forall v \in V.\ Q(v) = P(D^{-1}(v)).$

Prop. B.1. If $D{:}U \rightarrow V$ satisfies Condition 2 then D is a monotone bijection from U onto V.

Proof. D is a function from U to V, and Condition 2 requires that D^{-1} is a functon from V to U, so Conditon 2 requires that D is a bijection from U onto V. Next, assume that $x \leq y$, and let $Q_X = \lambda v \in V.$ (if $(v \geq D(x))$ then 1 else $\bot$). Then by Condition 2 there is a monotone function P_X such that $\forall v \in V.\ Q_X(v) = P_X(D^{-1}(v))$. Since D is a bijection, this is equivalent to $\forall u \in U.\ Q_X(D(u)) = P_X(u)$. Hence, $Q_X(D(y)) = P_X(y) \geq P_X(x) = Q_X(D(x)) = 1$ so $Q_X(D(y)) = 1$ and $D(y) \geq D(x)$. Thus D is monotone. ■

By Prop. B.1, Conditon 2 is too strong since it requires that every display in V is the display of some data object under D. Since U is a complete lattice it contains a maximal data object X (the least upper bound of all members of U). For all data objects

$u \in U$, $u \leq X$. Since D is monotone this implies $D(u) \leq D(X)$. We use the notation $\downarrow D(X)$ for the set of all displays less than $D(X)$. $\downarrow D(X)$ is a complete lattice and for all data objects $u \in U$, $D(u) \in \downarrow D(X)$. Hence we can replace V by $\downarrow D(X)$ in Condition 2 in order to not require that every $v \in V$ is the display of some data object. We modify Condition 2 as follows:

Condition 2'. $\forall Q \in MON(\downarrow D(X) \rightarrow \{\perp , 1\})$. $\exists P \in MON(U \rightarrow \{\perp , 1\})$.

$\forall v \in \downarrow D(X)$. $Q(v) = P(D^{-1}(v))$.

Def. A function $D:U \rightarrow V$ is a *display function* if it satisfies Conditions 1 and 2'.

The next two propositions demonstrate the consequences of this definition.

Prop. B.2. If $D:U \rightarrow V$ is a display function then:

(a) D is a bijective order embedding from U onto $\downarrow D(X)$

(b) $\forall v \in V$. $(\exists u' \in U. v \leq D(u') \Rightarrow \exists u \in U. v = D(u))$

(c) $\forall M \subseteq U$. $\bigvee D(M) = D(\bigvee M)$ and $\forall M \subseteq U$. $\bigwedge D(M) = D(\bigwedge M)$.

Proof. For part (1), D is a function from U to V, and Condition 2' requires that D^{-1} is a functon from $\downarrow D(X)$ to U, so D is a bijection from U onto $\downarrow D(X)$.

To show that D is an order embedding, assume that $D(x) \leq D(y)$, and let $P_X = \lambda u \in U$. (if $(u \geq x)$ then 1 else $\perp$). Then by Condition 1 there is a monotone function Q_X such that $\forall u \in U$. $Q_X(D(u)) = P_X(u)$. Hence, $P_X(y) = Q_X(D(y)) \geq Q_X(D(x)) = P_X(x) = 1$ so $P_X(y) = 1$ and $y \geq x$. Now assume that $x \leq y$, and let

$Q_X = \lambda v \in V.$ (if $(v \geq D(x))$ then 1 else $\perp$). Then by Condition 2' there is a monotone function P_X such that $\forall v \in V.\ Q_X(v) = P_X(D^{-1}(v))$. Since D is a bijection, this is equivalent to $\forall u \in U.\ Q_X(D(u)) = P_X(u)$. Hence, $Q_X(D(y)) = P_X(y) \geq P_X(x) = Q_X(D(x)) = 1$ so $Q_X(D(y)) = 1$ and $D(y) \geq D(x)$. Thus D is an order embedding.

For part (b), note that if $\exists u' \in U.\ v \leq D(u')$, then $v \leq D(X)$ and $v \in \downarrow D(X)$ so $\exists u \in U.\ v = D(u)$.

For part (c), $\forall m \in M.\ m \leq \bigvee M$ so $\forall m \in M.\ D(m) \leq D(\bigvee M)$ and so $\bigvee D(M) \leq D(\bigvee M)$. Thus, by part (2), $\exists u \in U.\ D(u) = \bigvee D(M)$, and $\forall m \in M.\ D(m) \leq D(u)$ so $\forall m \in M.\ m \leq u$ and thus $\bigvee M \leq u$. Therefore $D(\bigvee M) \leq D(u) = \bigvee D(M)$, and thus $D(\bigvee M) = \bigvee D(M)$.

Next, $\forall m \in M.\ \bigwedge M \leq m$ so $\forall m \in M.\ D(\bigwedge M) \leq D(m)$ and so $D(\bigwedge M) \leq \bigwedge D(M)$. For any $m \in M$, $\bigwedge D(M) \leq D(m)$, so, by part (2), $\exists u \in U.\ D(u) = \bigwedge D(M)$, and $\forall m \in M.\ D(u) \leq D(m)$ so $\forall m \in M.\ u \leq m$ and thus $u \leq \bigwedge M$. Therefore $\bigwedge D(M) = D(u) \leq D(\bigwedge M)$, and thus $D(\bigwedge M) = \bigwedge D(M)$. ■

As a corollary of Prop. B.2, next we show that display functions are lattice isomorphisms, and are continuous in the sense defined by Scott.

Prop. B.3. $D{:}U \to V$ is a display function if and only if it is a lattice isomorphism of U onto $\downarrow D(X)$, which is a sublattice of V. Furthermore, a display function D is continuous.

Proof. Assume $D{:}U \to V$ is a display function. For any $x, y \in U$, let $M = \{x, y\}$. Then, by Prop. B.2, $D(x \vee y) = D(x) \vee D(y)$ and $D(x \wedge y) = D(x) \wedge D(y)$, so D is a lattice homomorphism. Next, $a, b \in \downarrow D(X) \Rightarrow a, b \leq D(X) \Rightarrow a \vee b,\ a \wedge b \leq D(X) \Rightarrow$

$D(a \vee b)$, $D(a \wedge b) \in {\downarrow}D(X)$, so ${\downarrow}D(X)$ is a sublattice of V. By Prop. B.2, D is bijective, so it is a lattice isomorphism.

Assume $D{:}U \to {\downarrow}D(X)$ is a lattice isomorphism. If $x \leq y$ then $D(y) = D(x \vee y) = D(x) \vee D(y) \geq D(x)$. If $D(x) \leq D(y)$ then $y = D^{-1}(D(y)) = D^{-1}(D(x) \vee D(y)) = D^{-1}(D(x \vee y)) = x \vee y \geq x$. Thus D is an order embedding. Hence it is injective [that is $D(x) = D(y) \Rightarrow D(x) \leq D(y) \Rightarrow x \leq y$ and $D(x) = D(y) \Rightarrow D(y) \leq D(x) \Rightarrow y \leq x$, so $D(x) = D(y) \Rightarrow x = y$]

so D^{-1} is defined on $D(U) \subseteq V$. Given $P \in MON(U \to \{\perp , 1\})$, define $Q = \lambda v \in V. \bigvee\{P(D^{-1}(v')) \mid v' \leq v \;\&\; v' \in D(U)\}$. The set of v' such that $v' \leq v$ and $v' \in D(U)$ always includes $D(\perp)$, so Q is defined for all $v \in V$. Q is a function from V to $\{\perp , 1\}$, and Q is monotone since $v_1 \leq v_2 \Rightarrow \{v' \mid v' \leq v_1 \;\&\; v' \in D(U)\} \subseteq \{v' \mid v' \leq v_2 \;\&\; v' \in D(U)\}$. Then, for all $u \in U$,

$Q(D(u)) = \bigvee\{P(D^{-1}(v')) \mid v' \leq D(u) \;\&\; v' \in D(U)\} =$

$P(D^{-1}(D(u))) \vee \bigvee\{P(D^{-1}(v')) \mid v' < D(u) \;\&\; v' \in D(U)\} =$

[since P and D^{-1} are both monotone, $v' < D(u) \Rightarrow P(D^{-1}(v')) \leq P(D^{-1}(D(u)))$]

$P(D^{-1}(D(u))) = P(u)$.

This is equivalent to $P = Q \circ D$. Thus D satisfies Condition 1.

Given $Q \in MON(V \to \{\perp , 1\})$, define $P = \lambda u \in U.\ Q(D(u))$. P is a function from U to $\{\perp , 1\}$, and P is monotone since Q and D are monotone. Clearly $\forall u \in U.\ Q(D(u)) = P(u)$. Since D is a lattice isomorphism it is a bijection from U onto ${\downarrow}D(X)$ so this is equivalent to $\forall v \in {\downarrow}D(X).\ Q(v) = P(D^{-1}(v))$. Thus D satisfies Condition 2' and is a display function.

A display function D is an order embedding and thus monotone. For any directed set $M \subseteq U$, $\bigvee D(M) = D(\bigvee M)$ by Prop. B.2, so D is continuous. ■

Appendix C

Proofs for Section 3.2.2

Here we present the technical details for Section 3.2.2. Our lattices of data objects and of displays are defined in terms of scalar types. Each scalar type defines a value set, which may be either discrete or continuous, and which includes the undefined value $\perp$. We use the symbol **R** to denote the set of real numbers.

Def. A *discrete scalar s* defines a countable value set I_S that includes a least element $\perp$ and has discrete order. That is, $\forall x, y \in I_S. (x < y \Rightarrow (x = \perp \;\&\; y \neq \perp))$.

Def. A *continuous scalar s* defines a value set $I_S = \{\perp\} \cup \{[x, y] \mid x, y \in \mathbf{R} \;\&\; x \leq y\}$ (that is, the set of closed real intervals, plus $\perp$) with the order defined by: $\perp < [x, y]$ and $[x, y] \leq [u, v] \Leftrightarrow [u, v] \subseteq [x, y]$.

Prop. C.1. Discrete and continuous scalars are *cpos*. Discrete scalars are domains. However, a continuous scalar is not algebraic because its only compact element is $\perp$, and hence it is not a domain.

Proof. A discrete scalar *s* is clearly complete. To show that a continuous scalar *s* is complete, let *M* be a directed set in I_S. We need to show that $\bigvee M = \bigcap\{[u, v] \mid [u, v] \in M\}$ is an interval in I_S. Set $x = \max\{u \mid [u, v] \in M\}$ and $y = \min\{v \mid [u, v] \in M\}$. If $y < x$, set $a = x - y$, $y' = y + a / 3$ and $x' = x - a / 3$ so $y' < x'$. Then $\exists[u_1, v_1] \in M. v_1 \leq y'$ and $\exists[u_2, v_2] \in M. u_2 \geq x'$, so $[u_1, v_1] \cap [u_2, v_2] = \phi$. But

M directed implies that $\exists[u_3, v_3] \in M$. $[u_3, v_3] \subseteq [u_1, v_1] \cap [u_2, v_2]$. This is a contradiction, so x ≤ y and $[x, y] = \bigvee M$.

Let s be continuous and pick $[x, y] \in I_s$. To see if $[x, y]$ is compact, set $A_n = [x - 2^{-n}, y + 2^{-n}]$. Then [x, y] = $\bigvee_n A_n$ and $\{A_n\}$ is a directed set, but $\neg\exists A_n$. $[x, y] \leq A_n$ (that is, there is no interval A_n contained in $[x, y]$. Thus $\perp$ is the only compact element in I_s (for s continuous). ■

We define a tuple space as the cross product of a set of scalar value sets, and define a data lattice whose members are sets of tuples. Note that we use the notation $\mathsf{X}A$ for the cross product of members of a set A.

Def. Let S be a finite set of scalars, and let $X = \mathsf{X}\{I_s \mid s \in S\}$ be the set of tuples with an element from each I_s. Let a_s denote the s component of a tuple $a \in X$. Define an order relation on X by: for $a, b \in X$, $a \leq b$ if $\forall s \in S.\ a_s \leq b_s$.

Prop. C.2. Let $A \subseteq X$. If $b_s = \bigvee\{a_s \mid a \in A\}$ is defined for all $s \in S$ then $b = \bigvee A$. If $c_s = \bigwedge\{a_s \mid a \in A\}$ for all $s \in S$ then $c = \bigwedge A$ (that is, *sups* and *infs* over X are taken componentwise). Thus, X is a *cpo*.

Proof. $\forall s \in S.\ \forall a \in A.\ a_s \leq b_s$, so b is an upper bound for A. If e is another upper bound for A, then $\forall s \in S.\ b_s \leq e_s$ (since b_s is the least upper bound of $\{a_s \mid a \in A\}$). Thus, $b \leq e$, so b is the least upper bound of A. The argument that $c = \bigwedge A$ is similar.

Let $A \subseteq X$ be a directed set, and let $A_s = \{a_s \mid a \in A\}$. If $\{a_{is} \mid i\}$ is a finite subset of A_s, then $\{a_i \mid i\}$ is a finite subset of A, so $\exists e \in A.\ \forall i.\ a_i \leq e$. Then for each $s \in S$, $e_s \in A_s$ and $\forall i.\ a_{is} \leq e_s$, so A_s is a directed set, and thus $b_s = \bigvee A_s \in I_s$. As we just showed, $b = \bigvee A \in X$, so X is complete. ■

Def. We use $POWER(X) = \{A \mid A \subseteq X\}$ to denote the set of all subsets of X.

As explained in Section 3.2.2, $POWER(X)$ is not appropriate for a lattice structure, so we define equivalence classes on $POWER(X)$ using the Scott topology. The Scott topology defines open and closed sets as follows.

Def. A set $A \subseteq X$ is *open* if $\uparrow A \subseteq A$ and, for all directed subsets $C \subseteq X$, $\bigvee C \in A \Rightarrow C \cap A \neq \phi$.

Def. A set $A \subseteq X$ is *closed* if $\downarrow A \subseteq A$ and, for all directed subsets $C \subseteq A$, $\bigvee C \in A$. We use $CL(X)$ to denote the set of all closed subsets of X.

Def. Define a relation $\leq_R$ on $POWER(X)$ as follows: $A \leq_R B$ if for all open $C \subseteq X$, $A \cap C \neq \phi \Rightarrow B \cap C \neq \phi$. Also define a relation $\equiv_R$ on $POWER(X)$ as follows: $A \equiv_R B$ if $A \leq_R B$ and $B \leq_R A$.

Prop. C.3. The relation $\equiv_R$ is an equivalence relation.

Proof. Clearly $\forall A.A \leq_R A$ and thus $\forall A.A \equiv_R A$. And $A \equiv_R B \Leftrightarrow A \leq_R B \;\&\; B \leq_R A \Leftrightarrow B \equiv_R A$. If $A \leq_R B$ and $B \leq_R C$ then for all open $E \subseteq X$, $A \cap E \neq \phi \Rightarrow B \cap E \neq \phi$ and $B \cap E \neq \phi \Rightarrow C \cap E \neq \phi$, so $A \cap E \neq \phi \Rightarrow C \cap E \neq \phi$, and thus $A \leq_R C$. So $\equiv_R$ is reflexive, symmetric and transitive, and therefore an equivalence relation. ■

If $A \equiv_R B$ and $C \equiv_R D$, then $A \leq_R C \Leftrightarrow B \leq_R D$. Thus the equivalence classes of $\equiv_R$ are ordered by $\leq_R$. Now we show that the closed sets of the Scott topology can be used in place of the equivalence classes.

Def. Given an equivalence class E of the $\equiv_R$ relation, let $M_E = \bigcup E$.

Prop. C.4. Given an equivalence class E of the $\equiv_R$ relation, then $M_E \in E$.

Proof. Pick some $A \in E$. Then $A \subseteq M_E$ so $A \leq_R M_E$. For all open $C \subseteq X$, we have $M_E \cap C \neq \phi \Rightarrow \exists B \in E.\ B \cap C \neq \phi$ (since $M_E = \bigcup E$), but $B \cap C \neq \phi \Rightarrow A \cap C \neq \phi$ (since $B \leq_R A$). Thus $M_E \leq_R A$ and $M_E \equiv_R A$ so $M_E \in E$. ■

Prop. C.5. Given an equivalence class E of the $\equiv_R$ relation, then $M_E \in CL(X)$.

Proof. Given $a \in M_E$ and $b \leq a$, we need to show that $M_E \equiv_R M_E \cup \{b\}$ and hence that $b \in M_E$. Clearly $M_E \leq_R M_E \cup \{b\}$. For all open $C \subseteq X$, if $b \in C$ then $a \in C$ (since $b \leq a$) so $M_E \cap C \neq \phi$. Thus $M_E \cup \{b\} \leq_R M_E$ and $b \in M_E$.

Next, given a directed set $D \subseteq M_E$, let $b = \bigvee D$. Clearly $M_E \leq_R M_E \cup \{b\}$. For all open $C \subseteq X$, if $b \in C$ then $\exists c \in D.\ c \in C$ so $c \in M_E \cap C$. Thus $M_E \cup \{b\} \leq_R M_E$ and $b \in M_E$.

This shows that M_E is closed. ■

Prop. C.6. Given equivalence classes E and E' of the $\equiv_R$ relation, then $E \leq_R E' \Leftrightarrow M_E \subseteq M_{E'}$ and $E = E' \Leftrightarrow M_E = M_{E'}$. If $A \subseteq X$ is a closed set, then for some equivalence class E, $A = M_E$.

Proof. Note that $E \leq_R E' \Leftrightarrow M_E \leq_R M_{E'}$. If $M_E \subseteq M_{E'}$ then for all $C \subseteq X$ (whether C is open or not), $M_E \cap C \neq \phi \Rightarrow M_{E'} \cap C \neq \phi$ and thus $M_E \leq_R M_{E'}$. If

$\neg M_E \subseteq M_{E'}$ then there is $a \in M_E$ such that $a \notin M_{E'}$. The complement of $M_{E'}$, denoted $X \setminus M_{E'}$, is open, and $a \in M_E \cap (X \setminus M_{E'})$ but $M_{E'} \cap (X \setminus M_{E'}) = \phi$, so $\neg M_E \leq_R M_{E'}$.

$E = E' \Rightarrow M_E = \bigcup E = \bigcup E' = M_{E'}$. Conversely, $M_E = M_{E'} \Rightarrow M_E \leq_R M_{E'}$ & $M_{E'} \leq_R M_E \Rightarrow E \leq_R E'$ & $E \leq_R E' \Rightarrow E = E'$. Thus $E \leftrightarrow M_E$ is a one-to-one correspondence between closed sets and equivalence classes of $\equiv_R$.

If $A \subseteq X$ is a closed set, then A belongs to some equivalence class E so $A \subseteq M_E$ and $A \equiv_R M_E$. If $A \neq M_E$ then there is $a \in M_E$ such that $a \notin A$. $X \setminus A$ is open and $a \in M_E \cap (X \setminus A)$ but $A \cap (X \setminus A) = \phi$, so $\neg M_E \leq_R A$. This contradicts $A \equiv_R M_E$ so $A = M_E$. ■

The last proposition showed that there is a one to one correspondence between the equivalence classes of $\equiv_R$ and $CL(X)$. Next, we show that these closed sets obey the usual laws governing intersections and unions of closed sets in a topology.

Prop. C.7. If L is a set of closed subsets of X, then $\bigcap L$ is closed. If L is finite, then $\bigcup L$ is closed. Furthermore, for all $x \in X$, $\downarrow x \in CL(X)$.

Proof. If $x \in \bigcap L$ and $y \leq x$, then for all $A \in L$, $x \in A$ and $\downarrow A \subseteq A$, so $y \in A$ and so $y \in \bigcap L$. Thus $\downarrow \bigcap L \subseteq \bigcap L$. If C is a directed subset $C \subseteq \bigcap L$, then for all $A \in L$, $C \subseteq A$ and $\bigvee C \in A$. Thus $\bigvee C \in \bigcap L$ and $\bigcap L$ is closed.

Now assume L is finite. If $x \in \bigcup L$ and $y \leq x$, then for some $A \in L$, $x \in A$ and $\downarrow A \subseteq A$, so $y \in A$ and so $y \in \bigcup L$. Thus $\downarrow \bigcup L \subseteq \bigcup L$. Let C be a directed subset $C \subseteq \bigcup L$ and assume that $\bigvee C \notin \bigcup L$. Then $\forall A \in L.\ \bigvee C \notin A$ so, since all $A \in L$ are closed, $\forall A \in L.\ \neg C \subseteq A$. Thus $\forall A \in L.\ \exists c_A \in C.\ c_A \notin A$. Now, $\{c_A \mid A \in L\}$ is finite, so $\exists c \in C.\ \forall A \in L.\ c_A \leq c$. But $\forall A \in L.\ c_A \notin A \Rightarrow c \notin$ A (since $A \in L$ are down sets), so $c \notin \bigcup L$. This contradicts $C \subseteq \bigcup L$ so we must have $\bigvee C \in \bigcup L$. Thus $\bigcup L$ is closed.

Clearly $\downarrow(\downarrow x) \subseteq \downarrow x$. If $C \subseteq \downarrow x$ is a directed set (or any subset of $\downarrow x$), then $\forall c \in C.\ c \leq x$ so $\bigvee C \leq x$ and thus $\bigvee C \in \downarrow x$. Therefore $\downarrow x$ is closed. ■

Now we show that the equivalence classes of the $\equiv_R$ relation, and equivalently $CL(X)$, form a complete lattice.

Prop. C.8. If W is a set of equivalence classes of the $\equiv_R$ relation, and then $\bigwedge W$ is defined and $\bigwedge W = E$ such that $M_E = \bigcap\{M_w \mid w \in W\}$. Similarly, $\bigvee W$ is defined and $\bigvee W = E$ such that M_E is the smallest closed set containing $\bigcup\{M_w \mid w \in W\}$. Thus the equivalence classes of the $\equiv_R$ relation form a complete lattice, and equivalently $CL(X)$ is a complete lattice. If W is finite and $E = \bigvee W$, then $M_E = \bigcup\{M_w \mid w \in W\}$.

Proof. By Prop. C.7, $\bigcap\{M_w \mid w \in W\}$ is closed and, by Prop. C.6, must be M_E for some equivalence class E. Now, $\forall w \in W.\ M_E \subseteq M_w$ so $\forall w \in W.\ M_E \leq_R M_w$ and $\forall w \in W.\ E \leq_R w$. If E' is an equivalence class such that $\forall w \in W.\ E' \leq_R w$, then $\forall w \in W.\ M_{E'} \subseteq M_w$, so $M_{E'} \subseteq M_E$ and $E' \leq_R E$. Thus $E = \bigvee W$.

By Prop. C.7, the intersection of all closed sets containing $\bigcup\{M_w \mid w \in W\}$ must be a closed set and, by Prop. C.6, must be M_E for some equivalence class E. Now, $\forall w \in W.\ M_w \subseteq M_E$ so $\forall w \in W.\ M_w \leq_R M_E$ and $\forall w \in W.\ w \leq_R E$.
If E' is an equivalence class such that $\forall w \in W.\ w \leq_R E'$, then $\forall w \in W.\ M_w \subseteq M_{E'}$, so $M_{E'}$ contains $\bigcup\{M_w \mid w \in W\}$. Thus $M_E \subseteq M_{E'}$ and $E \leq_R E'$. Therefore $E = \bigvee W$.

If W is finite, then $\bigcup\{M_w \mid w \in W\}$ is closed and equal to M_E, where $E = \bigvee W$. ■

Now we prove two propositions that will be useful for determining when sets of tuples are closed.

Prop. C.9. If $a \in X$, $B \subseteq X$ and $a \le \bigvee B$ then $a = \bigvee\{a \wedge b \mid b \in B\}$.

Proof. Let a_S and b_S denote the tuple components of a and b. The order relation, *sups* and *infs* of a cross product are taken componentwise, so it is sufficient to prove the proposition for each tuple component. That is, we will show that
$\forall s \in S.\ a_S \le \bigvee\{a_S \wedge b_S \mid b \in B\}$.

For discrete s, I_S has the discrete order. If $\bigvee\{b_S \mid b \in B\} = \bot$ then $a_S = \bot$ and $\forall s \in S.\ b_S = \bot$, and the conclusion is clearly true. Otherwise, let $c_S = \bigvee\{b_S \mid b \in B\}$. Then $\forall b \in B.\ (b_S = \bot$ or $b_S = c_S)$. If $a_S = \bot$ then $\forall b \in B.\ a_S \wedge b_S = \bot$ and
$a_S = \bot = \bigvee\{a_S \wedge b_S \mid b \in B\}$. Otherwise $a_S = c_S$ and $\forall b \in B.\ a_S \wedge b_S = b_S$ and
$a_S = c_S = \bigvee\{b_S \mid b \in B\} = \bigvee\{a_S \wedge b_S \mid b \in B\}$.

For continuous s, the members of I_S are real intervals, or are $\bot$. Let $a_S = [x_S, y_S]$ and $b_S = [x(b_S), y(b_S)]$, where we use $x = -\infty$ and $y = +\infty$ for $a_S = \bot$ or $b_S = \bot$. The order relation on I_S corresponds to the inverse of interval containment, *sup* corresponds to intersection of intervals, and *inf* corresponds to the smallest interval containing the union of intervals. First, note that $\forall b \in B.\ a \wedge b \le a$ and thus $\bigvee\{a \wedge b \mid b \in B\} \le a$. So, it is only necessary to show that $a \le \bigvee\{a \wedge b \mid b \in B\}$, or, in other words, that the intersection of the intervals $[\min\{x_S, x(b_S)\}, \max\{y_S, y(b_S)\}]$ for all $b \in B$ is contained in the interval $[x_S, y_S]$. This intersection of intervals is
$[c, d] = [\max\{\min\{x_S, x(b_S)\} \mid b \in B\}, \min\{\max\{y_S, y(b_S)\} \mid b \in B\}]$. Now,
$a_S \le \bigvee\{b_S \mid b \in B\}$ says that $x_S \le \max\{x(b_S) \mid b \in B\}$ and $\min\{x(b_S) \mid b \in B\} \le y_S$. So for at least one $b \in B$, $x_S \le x(b_S)$ and $\min\{x_S, x(b_S)\} = x_S$, and thus
$c = \max\{\min\{x_S, x(b_S)\} \mid b \in B\} \ge x_S$. Similarly $d \le y_S$, and so $[c, d] \subseteq [x_S, y_S]$, showing the needed containment. ■

Prop. C.10. If $Y \subseteq CL(X)$ then $B = \{\bigvee M \mid M \subseteq \bigcup Y \ \&\ M \text{ directed}\}$ is closed.

Proof. First, we show that B is a down set. By Prop. C.9, $a \leq \bigvee M \Rightarrow a = \bigvee\{a \wedge m \mid m \in M\}$, so we need to show that $\bigvee\{a \wedge m \mid m \in M\}$ is directed when M is. Given a finite set $\{a \wedge b_i \mid b_i \in M\}$ there is c in M such that $\forall i.\ b_i \leq c$, and thus $\bigvee_i b_i \leq c$. Now $\forall i.\ b_i \leq \bigvee_i b_i \Rightarrow \forall i.\ a \wedge b_i \leq a \wedge \bigvee_i b_i \Rightarrow \vee_i(a \wedge b_i) \leq a \wedge \bigvee_i b_i \leq a \wedge c$. However $a \wedge c \in \bigvee\{a \wedge m \mid m \in M\}$, so $\{a \wedge m \mid m \in M\}$ is directed, $a \in B$ and B is a down set.

Next, we show that B is closed under *sups*. Let M be a directed subset of B and we will show that $a = \bigvee M \in B$. For each $m \in M$ there is a directed set $Q(m) \subseteq \bigcup Y$ such that $m = \bigvee Q(m)$. Define $Q' = \bigcup\{Q(m) \mid m \in M\}$ and $Q = \{\bigvee C \mid C \subseteq Q' \ \&\ C \text{ finite}\}$. Note that $\bigvee Q'$ exists (and $= a$) so $\bigvee C$ exists. For each finite $C \subseteq Q'$, each $c \in C$ belongs to a member of Y. Thus C is a subset of a finite union of members of Y, which is a closed set, so $\bigvee C$ must belong to this same closed set and therefore belongs to $\bigcup Y$. Thus $Q \subseteq \bigcup Y$. Pick a finite set $\{q_i\} \subseteq Q$. Each q_i is the *sup* of a finite subset $C_i \subseteq Q'$, and $\bigvee_i q_i$ is the *sup* of the finite subset $\bigcup_i C_i$ of Q'. Thus $\bigvee_i q_i \in Q$ so Q is a directed subset of $\bigcup Y$ with $a = \bigvee Q = \bigvee Q'$, so a is a member of B. Thus B is closed under sups, and is a closed set. ■

Appendix D

Proofs for Section 3.2.3

Here we present the technical details for Section 3.2.3.

Def. A set T of *data types* can be defined from the set S of scalars. Two functions, SC and DOM are defined with T, such that $\forall t \in T.\ SC(t) \subseteq S\ \&\ DOM(t) \subseteq S$. T, SC and DOM are defined as follows:

(D.1) $s \in S \Rightarrow s \in T$ (that is, $S \subset T$)

$SC(s) = \{s\}$

$DOM(s) = \phi$.

(D.2) (for $i = 1,...,n.\ t_i \in T$) & ($i \neq j \Rightarrow SC(t_i) \cap SC(t_j) = \phi$) $\Rightarrow struct\{t_1;...;t_n\} \in T$

$SC(struct\{t_1;...;t_n\}) = \bigcup_i SC(t_i)$

$DOM(struct\{t_1;...;t_n\}) = \bigcup_i DOM(t_i)$

(D.3) $w \in S\ \&\ r \in T\ \&\ w \notin SC(r) \Rightarrow (array\ [w]\ of\ r) \in T$

$SC((array\ [w]\ of\ r)) = \{w\} \cup SC(r)$

$DOM((array\ [w]\ of\ r)) = \{w\} \cup DOM(r)$

The type $struct\{t_1;...;t_n\}$ is a *tuple* with *element* types t_i, and the type ($array$ [w] of r) is an *array* with *domain* type w and *range* type r. $SC(t)$ is the set of scalars occurring in t, and $DOM(t)$ is the set of scalars occurring as array domains in t. Note that each scalar in S may occur at most once in a type in T.

Def. For each scalar $s \in S$, define a countable set $H_s \subseteq I_s$ such that for all $a, b \in H_s$, $a \wedge b \in H_s$, $a \vee b \in I_s \Rightarrow a \vee b \in H_s$, and such that $\forall a \in I_s. \exists A \subseteq H_s. a = \bigvee A$ (that is, H_s is closed under *infs* and *sups*, and any member of I_s is a *sup* of a set of members of H_s). For discrete s this implies that $H_s = I_s$ (recall that we defined discrete scalars as having countable value sets). Also note that, for continuous s, H_s cannot be a *cpo*.

Def. Given a scalar w, let

$FIN(H_w) = \{A \subseteq H_w \backslash \{\bot\} \mid A \text{ finite } \& \ \forall a, b \in A. \neg(a \leq b)\}$.

Def. Extend the definition of H_t to $t \in T$ by:

(D.4) $t = struct\{t_1;...;t_n\} \Rightarrow H_t = H_{t_1} \times ... \times H_{t_n}$

(D.5) $t = (array\ [w]\ of\ r) \Rightarrow H_t = \bigcup\{(A \rightarrow H_r) \mid A \in FIN(H_w)\}$

Def. Define an embedding $E_t{:}H_t \rightarrow U$ by:

(D.6) $t \in S \Rightarrow E_t(a) = \downarrow(\bot,...,a,...,\bot)$

(D.7) $t = struct\{t_1;...;t_n\} \Rightarrow E_t((a_1,...,a_n)) = \{b_1 \vee ... \vee b_n \mid \forall i.\ b_i \in E_{t_i}(a_i)\}$

(D.8) $t = (array\ [w]\ of\ r) \Rightarrow$

$$[a \in (A \to H_r) \Rightarrow E_t(a) = \{b \vee c \mid x \in A \ \&\ b \in E_w(x) \ \&\ c \in E_r(a(x))\}]$$

The notation $\downarrow(\perp,...,a,...,\perp)$ in Eq. (D.6) indicates the closed set of all tuples less than $(\perp,...,a,...,\perp)$. As we will show in Prop. D.1, for all $a \in H_t$ and for all $b \in E_t(a)$, $b_s = \perp$ unless $s \in SC(t)$. Thus $b_1 \vee ... \vee b_n$ in Eq. (D.7) is the tuple that merges the non-$\perp$ components of the tuples b_1, ..., b_n., since the types t_i in Eq. (D.7) are defined from disjoint sets of scalars. Similarly, $b \vee c$ in Eq. (D.8) is the tuple that merges the non-$\perp$ components of the tuples b and c, since the scalar w does not occur in the type r. Prop. D.2 will show that E_t does indeed map members of H_t to members of U.

Def. For $t \in T$ define $F_t = E_t(H_t)$.

Prop. D.1. Given $t \in T$ and $A \in F_t$, for all tuples $b \in A$, $\forall s \in S.\ (s \notin SC(t) \Rightarrow b_s = \perp)$.

Proof. We prove this by induction on the structure of t. This is clearly true for $t \in S$. For $t = struct\{t_1;...;t_n\}$ pick $b = b_1 \vee ... \vee b_n \in A \in F_t$, where $b_i \in B_i \in F_{t_i}$. Then $b_s = b_{1s} \vee ... \vee b_{ns}$. By induction, $\forall i.\ \forall s.\ s \notin SC(t_i) \Rightarrow b_{is} = \perp$, so $\forall s.\ (\forall i.\ s \notin SC(t_i)) \Rightarrow b_s = \perp$, and so $\forall s.\ s \notin \bigcup_i SC(t_i) \Rightarrow b_s = \perp$. But $SC(t) = \bigcup_i SC(t_i)$.

For $t = (array\ [w]\ of\ r)$ pick $a = b \vee c \in A \in F_t$, where $b \in B \in F_w$ and $c \in C \in F_r$. Then $a_s = b_s \vee c_s$. By induction, $s \neq w \Rightarrow b_s = \perp$ and $s \notin SC(r)) \Rightarrow c_s = \perp$, so $\forall s.\ s \notin \{w\} \cup SC(r) \Rightarrow b_s = \perp$. But $SC(t) = \{w\} \cup SC(r)$. ■

The following propositions show that E_t maps members of H_t to closed sets, and that this mapping is injective.

Prop. D.2. For all $a \in H_t$, $E_t(a)$ is a closed set.

Proof. We prove this by induction on the structure of t. For $t \in S$, $E_t(a) = \downarrow(\bot,...,a,...,\bot)$ is closed, by Prop. C.7. For $t = struct\{t_1;...;t_n\}$, we need to show that $E_t(a) = \{b_1 \vee ... \vee b_n \mid \forall i.\ b_i \in E_{t_i}(a_i)\}$ is closed, where $a = (a_1,...,a_n)$. To show that $E_t(a)$ is a down-set, pick $b \leq b_1 \vee ... \vee b_n \in E_t(a)$. Then $\forall i.\ b \wedge b_i \leq b_i$ and hence $\forall i.\ b \wedge b_i \in E_{t_i}(a_i)$ (since these are down sets). Thus, by Prop. C.9, $b = (b \wedge b_1) \vee ... \vee (b \wedge b_n) \in E_t(a)$. To show that $E_t(a)$ is closed under *sups* of directed sets, pick a directed set $C \subseteq E_t(a)$ and for all $c \in C$ let $c = b_1(c) \vee ... \vee b_n(c)$ where $\forall i.\ b_i(c) \in E_{t_i}(a_i)$. We need to show that $C_i = \{b_i(c) \mid c \in C\}$ is a directed set. Pick a finite subset $\{b_i(c_j) \mid j\} \subseteq C_i$. Since C is directed, there is $m \in C$ such that $\forall j.\ c_j \leq m$. Note that $m = b_1(m) \vee ... \vee b_n(m)$ where $\forall i.\ b_i(m) \in C_i$. Since the t_i have disjoint sets of non-$\bot$ components, $\forall i.\ \forall j.\ b_i(c_j) \leq b_i(m)$. Thus C_i is directed, and $\bigvee C_i \in E_{t_i}(a_i)$. Hence $\bigvee C = \bigvee C_1 \vee ... \vee \bigvee C_n \in E_t(a)$, and thus $E_t(a)$ is closed under *sups* of directed sets.

For $t = (array\ [w]\ of\ r)$, we need to show that $E_t(a) = \{b \vee c \mid x \in A\ \&\ b \in E_w(x)\ \&\ c \in E_r(a(x))\}$ is closed, where $a \in (A \to H_r)$. Define $E_t(a)_x = \{b \vee c \mid b \in E_w(x)\ \&\ c \in E_r(a(x))\}$. Note that $E_t(a)_x = E_{struct\{w;r\}}((a, a(x)))$ [where $struct\{w; r\}$ is a tuple type and $(a, a(x)) \in H_{struct\{w;r\}}$] and thus, by the argument above for tuple types, $E_t(a)_x$ is closed. Also note that $E_t(a) = \bigcup\{E_t(a)_x \mid x \in A\}$. However, A is finite, so $E_t(a)$ is a union of a finite number of closed sets, and thus is itself closed. ■

Prop. D.3. The embedding $E_t : H_t \to U$ is injective.

Proof. We prove this by induction on the structure of t.

Let t be a scalar and $a \neq b$. Then $\neg(a \leq b)$ or $\neg(b \leq a)$. Assume without loss of generality that $\neg(a \leq b)$. Then $(\perp,...,a,...,\perp) \in \downarrow(\perp,...,a,...,\perp) = E_t(a)$ but $(\perp,...,a,...,\perp) \notin \downarrow(\perp,...,b,...,\perp) = E_t(b)$, so $E_t(a) \neq E_t(b)$.

Let $t = struct\{t_1;...;t_n\}$ and $a = (a_1,...,a_n) \neq (b_1,...,b_n) = b$. Then $\exists k.\ a_k \neq b_k$ and, by the inductive hypothesis, $E_{t_k}(a_k) \neq E_{t_k}(b_k)$. Assume without loss of generality that $\exists c_k \in E_{t_k}(a_k).\ c_k \notin E_{t_k}(b_k)$, and for all $i \neq k$ pick $c_i \in E_{t_i}(a_i)$. Then $c_1 \vee ... \vee c_n \in E_t((a_1,...,a_n))$, but, since $c_k \notin E_{t_k}(b_k)$ and since $\forall s \in S. \forall i \neq k.\ c_{ks} \neq \perp \Rightarrow c_{is} = \perp$, $c_1 \vee ... \vee c_n \notin E_t((b_1,...,b_n))$. Thus $E_t((a_1,...,a_n)) \neq E_t((b_1,...,b_n))$.

Let $t = (array\ [w]\ of\ r)$ and $a \neq b$ where $a \in (A \to H_r)$ and $b \in (B \to H_r)$. Then either $A \neq B$ or $A = B\ \&\ \exists x \in A.\ a(x) \neq b(x)$. In the first case (that is, $A \neq B$), assume without loss of generality that $\exists x \in A.\ x \notin B$. If $\exists y \in B.\ x \leq y$ then $\neg\exists z \in A.\ y \leq z$ (otherwise $x \in A\ \&\ z \in A\ \&\ x \leq z$). Thus either $\exists x \in A.\ \neg(\exists y \in B.\ x \leq y)$ or $\exists y \in B.\ \neg(\exists z \in A.\ y \leq z)$. Assume without loss of generality that $\exists x \in A.\ \neg(\exists y \in B.\ x \leq y)$. Then $e = (\perp,...,x,...,\perp) \in E_w(x)$ and $\neg(\exists y \in B.\ e \in E_w(y))$. Pick $f \in E_r(a(x))$. Then $e \vee f \in E_t(a)$ but $e \vee f \notin E_t(b)$, so $E_t(a) \neq E_t(b)$. In the second case (that is, $A = B\ \&\ \exists x \in A.\ a(x) \neq b(x)$), by the inductive hypothesis, $E_r(a(x)) \neq E_r(b(x))$. Assume without loss of generality that $\exists x \in A.\ \exists f \in E_r(a(x)).\ f \notin E_r(b(x))$. Pick $e \in E_w(x)$. Then $e \vee f \in E_t(a)$ but $e \vee f \notin E_t(b)$, so $E_t(a) \neq E_t(b)$. ∎

Because $E_t : H_t \to U$ is injective, we can define an order relation between the members of H_t simply by assuming that E_t is an order embedding. If E_t were not injective, it would map a pair of members of H_t to the same member of U, and induce an anti-symmetric relation on H_t.

Def. Given a, $b \in H_t$, we say that $a \leq b$ if and only if $E_t(a) \leq E_t(b)$.

The order that E_t induces on H_t has a simple and intuitive structure, as the following proposition shows.

Prop. D.4. If t is a scalar and a, $b \in H_t$ then $E_t(a) \leq E_t(b)$ if and only if $a \leq b$ in I_t. If $t = struct\{t_1;...;t_n\}$ then $E_t((a_1,...,a_n)) \leq E_t((b_1,...,b_n))$ if and only if $\forall i.\ E_{t_i}(a_i) \leq E_{t_i}(b_i)$ (that is, the order relation between tuples is defined element-wise). If $t = (array\ [w]\ of\ r)$, if a, $b \in H_t$ and if $a \in (A \to H_r)$ and $b \in (B \to H_r)$, then $E_t(a) \leq E_t(b)$ if and only if $\forall x \in A.\ E_r(a(x)) \leq \bigvee\{E_r(b(y)) \mid y \in B\ \&\ E_w(x) \leq E_w(y)\}$ (that is, an array a is less than an array b if the embedding of the value of a at any sample x is less than the *sup* of the embeddings of the set of values of b at its samples greater than x).

Proof. Recall that members of U are closed sets ordered by set inclusion, so $E_t(a) \leq E_t(b) \Leftrightarrow E_t(a) \subseteq E_t(b)$. Let t be a scalar. If $a \leq b$ in I_t then

$$E_t(a) = \downarrow(\bot,...,a,...,\bot) = \{(\bot,...,c,...,\bot) \mid c \leq a\} \subseteq$$

$$\{(\bot,...,c,...,\bot) \mid c \leq b\} = \downarrow(\bot,...,b,...,\bot) = E_t(b).$$

Now assume that $E_t(a) \leq E_t(b)$. Then

$$E_t(a) = \downarrow(\bot,...,a,...,\bot) = \{(\bot,...,c,...,\bot) \mid c \leq a\} \subseteq$$

$$\{(\bot,...,c,...,\bot) \mid c \leq b\} = \downarrow(\bot,...,b,...,\bot) = E_t(b)$$

so $(\bot,...,a,...,\bot) \in \{(\bot,...,c,...,\bot) \mid c \leq b\}$ so $a \leq b$ in I_t.

Let $t = struct\{t_1;...;t_n\}$. If $\forall i. E_{t_i}(a_i) \subseteq E_{t_i}(b_i)$ then

$E_t((a_1,...,a_n)) = \{c_1 \vee ... \vee c_n \mid \forall i.\ c_i \in E_{t_i}(a_i)\} \subseteq$

$\{c_1 \vee ... \vee c_n \mid \forall i.\ c_i \in E_{t_i}(b_i)\} = E_t((b_1,...,b_n))$.

Now assume that $E_t((a_1,...,a_n)) \leq E_t((b_1,...,b_n))$. Then

$E_t((a_1,...,a_n)) = \{c_1 \vee ... \vee c_n \mid \forall i.\ c_i \in E_{t_i}(a_i)\} \subseteq$

$\{c_1 \vee ... \vee c_n \mid \forall i.\ c_i \in E_{t_i}(b_i)\} = E_t((b_1,...,b_n))$.

[Parenthetical argument: assume that $c_k \notin E_{t_k}(b_k)$, $c_i \in E_{t_i}(a_i)$ for $i \neq k$, and $c_1 \vee ... \vee c_n \in E_t((b_1,...,b_n))$. Then there are $d_i \in E_{t_i}(b_i)$ such that $c_1 \vee ... \vee c_n = d_1 \vee ... \vee d_n$. However, $i \neq j \Rightarrow SC(t_i) \cap SC(t_j) = \phi$ so, by Prop. D.1, $d_{is} = \bot$ for $s \in SC(t_k)$ and $i \neq k$. Thus $c_{ks} = d_{ks}$ for $s \in SC(t_k)$ and so $c_k = d_k$. This is impossible, so $c_1 \vee ... \vee c_n \in E_t((b_1,...,b_n))$ and $c_i \in E_{t_i}(a_i)$ for $i \neq k \Rightarrow c_k \in E_{t_k}(b_k)$.]

Thus $\forall i.\ (c_i \in E_{t_i}(a_i) \Rightarrow c_i \in E_{t_i}(b_i))$, or in other words, $\forall i. E_{t_i}(a_i) \subseteq E_{t_i}(b_i)$.

Let $t = (array\ [w]\ of\ r)$, $a, b \in H_t$ and $a \in (A \to H_r)$ and $b \in (B \to H_r)$. Assume that $\forall x \in A.\ E_r(a(x)) \leq \bigvee\{E_r(b(y)) \mid y \in B\ \&\ E_w(x) \leq E_w(y)\}$. Then $\forall x \in A.\ E_r(a(x)) \subseteq \bigcup\{E_r(b(y)) \mid y \in B\ \&\ E_w(x) \leq E_w(y)\}$.

$E_t(a) = \{e \vee f \mid x \in A\ \&\ e \in E_w(x)\ \&\ f \in E_r(a(x))\} =$

$\bigcup\{\{e \vee f \mid e \in E_w(x)\ \&\ f \in E_r(a(x))\} \mid x \in A\}$.

Now, $f \in E_r(a(x)) \Rightarrow \exists y \in B.\ E_w(x) \leq E_w(y)\ \&\ f \in E_r(b(y))$ and $e \in E_w(x)\ \&\ E_w(x) \leq E_w(y) \Rightarrow e \in E_w(y)$, so (continuing the chain)

$$\bigcup\{\{e \vee f \mid e\in E_w(x) \;\&\; f\in E_r(a(x))\} \mid x\in A\} \subseteq$$
$$\bigcup\{\{e \vee f \mid e\in E_w(y) \;\&\; f\in E_r(b(y)) \;\&\; E_w(x) \leq E_w(y) \;\&\; y \in B\} \mid x\in A\} \subseteq$$
$$\bigcup\{\{e \vee f \mid e\in E_w(y) \;\&\; f\in E_r(b(y))\} \mid y \in B\} =$$
$$\{e \vee f \mid y\in B \;\&\; e\in E_w(y) \;\&\; f\in E_r(b(y))\} = E_t(b).$$

Thus $E_t(a) \leq E_t(b)$.

Now assume that $E_t(a) \leq E_t(b)$. That is,

$$E_t(a) = \{e \vee f \mid x\in A \;\&\; e\in E_w(x) \;\&\; f\in E_r(a(x))\} \subseteq$$
$$\bigcup\{\{e \vee f \mid e\in E_w(y) \;\&\; f\in E_r(b(y))\} \mid y \in B\} = E_t(b).$$

Since $w \notin SC(r)$, $e\in E_w(x) \;\&\; f\in E_r(a(x)) \;\&\; e\vee f \in E_t(b) \Rightarrow \exists y\in B.\ e\in E_w(y) \;\&\; f\in E_r(b(y))$ [this is a result of the parenthetical argument in the tuple case of this proof]. Pick $x \in A$ and $f \in E_r(a(x))$, and define $e = (\bot,...,x,...,\bot)$. Then $\exists y\in B.\ e\in E_w(y) \;\&\; f\in E_r(b(y))$. Now $e \in E_w(y) \Rightarrow x \leq y \Rightarrow E_w(x) \leq E_w(y)$ so $f \in \bigcup\{E_r(b(y)) \mid y \in B \;\&\; E_w(x) \leq E_w(y)\} = \bigvee\{E_r(b(y)) \mid y \in B \;\&\; E_w(x) \leq E_w(y)\}$. Thus
$\forall x \in A.\ E_r(a(x)) \leq \bigvee\{E_r(b(y)) \mid y \in B \;\&\; E_w(x) \leq E_w(y)\}$. ■